Функциональный интеграл для многочастичной КМ

$$\hat{H} = \int \left(\frac{\nabla \hat{\psi}^+ \nabla \hat{\psi}}{2m} + U(x)\hat{\psi}^+(x)\hat{\psi}(x) \right) dx +$$
$$+ \frac{1}{2} \int d\vec{x}_1 d\vec{x}_2' \, \hat{\psi}^+(x)\, \hat{\psi}^+(x')\, U(x-x')\, \hat{\psi}(x')\, \hat{\psi}(x)$$
$$\Leftrightarrow \hat{H}_N = \sum_i \left(\frac{\hat{p}_i}{2m} + U(\hat{x}_i) \right) + \frac{1}{2} \sum_{i \neq j} U(x_i - x_j)$$

Базис когерентных состояний:

БОЗОНЫ:

1) $[\hat{a}_i, \hat{a}_j^+] = \delta_{ij}$ $\quad \varphi \in \mathbb{C}$

$\hat{a}|\varphi\rangle = \varphi|\varphi\rangle$

$|\varphi\rangle = e^{\varphi \hat{a}^+}|0\rangle = \sum_n \frac{\varphi^n}{\sqrt{n!}}|n\rangle$

$\langle\theta|\varphi\rangle = e^{\bar{\theta}\varphi}$

$\hat{1} = \sum_n |n\rangle\langle n| = \int \frac{d(\mathrm{Re}\,\varphi)\, d(\mathrm{Im}\,\varphi)}{\pi} e^{-|\varphi|^2} |\varphi\rangle\langle\varphi|$

2) Фермионы

$\{\hat{a}_i, \hat{a}_j^+\} = \delta_{ij}$

$\hat{a}_i|\eta_i\rangle = \eta_i|\eta_i\rangle$ $\qquad \overset{0}{\overset{\parallel}{\{\hat{a}_i, \hat{a}_j\}}}|\eta_i, \eta_j\rangle = \underbrace{(\eta_i\eta_j + \eta_j\eta_i)}_{\text{Грассманова алгебра}}|\eta_i, \eta_j\rangle$

$\hat{a}_j|\eta_j\rangle = \eta_j|\eta_j\rangle$ $\qquad \hat{a}^2|\psi\rangle = 0$

Грассмановы числа

1) $\eta_i\eta_j + \eta_j\eta_i = 0$

$\Rightarrow \eta^2 = 0$

2) $(\eta_1\eta_2)\eta_3 = \eta_1(\eta_2\eta_3)$

$\eta_1(\eta_2 + \eta_3) = \eta_1\eta_2 + \eta_1\eta_3$

3) $\{\eta_i\}_{i=1}^N \Leftrightarrow$ выр-я: $a + \sum_i a_i\eta_i + \sum_{i \neq j} a_{ij}\eta_i\eta_j + \dots + \dots + a_{12\dots N}\eta_1\dots\eta_N$

$\Leftrightarrow \mathbb{C}_{2^N}$

4) $f(\overset{\in\mathbb{C}}{x} + \eta) \underset{def}{=} f(x) + f'(x)\cdot\eta$

5) $\frac{\partial}{\partial\eta_i}\eta_j = \delta_{ij}$ $\qquad \frac{\partial}{\partial\eta}(\xi\eta) = -\frac{\partial}{\partial\eta}(\eta\xi) = -\xi$

6) $\int d\eta_i \cdot \eta_j = \delta_{ij}$

$\int d\eta\, d\xi \cdot \eta\xi = -\int d\xi\, d\eta\, \eta\xi = -1$

7) "Комплексное сопряж."

$\eta \to \bar{\eta}$ независима $\quad \bar{\bar{\eta}} = \eta$

8) Грассманов Гаусс:

$\int \prod_i (d\bar{\eta}_i d\eta_i) \exp(-\sum_{ij} \bar{\eta}_i L_{ij} \eta_j) = \det \hat{L}$

Диаг: $\int \prod_i d\bar{\eta}_i d\eta_i \exp(-\sum_i \lambda_i \bar{\eta}_i \eta_i)$

$\prod_i (1 - \lambda_i \bar{\eta}_i \eta_i) =$

$= \int \prod_i \left[d\bar{\eta}_i d\eta_i (\cancel{1} - \lambda_i \bar{\eta}_i \eta_i) \right] = \prod_i \lambda_i = \det \hat{L}$

Грассмановы ког. сост.

(Fermi): $\hat{a}|\eta\rangle = \eta|\eta\rangle$

$|\eta\rangle = e^{\eta\hat{a}^+}|0\rangle = (1 + \eta a^+)|0\rangle = |0\rangle + \eta \cdot |1\rangle$

$\langle\eta|\xi\rangle = (\langle 0| + \bar{\eta}\cdot\langle 1|)(|0\rangle + \xi|1\rangle) = 1 + \bar{\eta}\xi = e^{\bar{\eta}\xi}$

$\hat{1} = |0\rangle\langle 0| + |1\rangle\langle 1| = \int d\bar{\eta} d\eta\, e^{-\bar{\eta}\eta} |\eta\rangle\langle\eta| =$

$= \int d\bar{\eta} d\eta\, (1 - \bar{\eta}\eta)(|0\rangle + \eta|1\rangle)(\langle 0| + \bar{\eta}\langle 1|) = |0\rangle\langle 0| + |1\rangle\langle 1|$

Path integral для КМ

1) $\hat{H} = \omega\hat{a}^+\hat{a}$ Нормально упор. гамильт.

$\dots \langle\varphi_i|(1 - i\hat{H}\epsilon)|\varphi\rangle = e^{\bar{\varphi}'\varphi} - iH(\bar{\varphi}', \varphi)\overset{\epsilon\to 0}{\epsilon} e^{\bar{\varphi}'\varphi} = e^{\bar{\varphi}'\varphi} \cdot e^{-iH(\bar{\varphi}', \varphi)\epsilon}$

$G_R(\varphi, \varphi', T>0) = \langle\varphi'|e^{-i\hat{H}T}|\varphi\rangle = \langle\varphi'|e^{-i\hat{H}\epsilon}\dots e^{-i\hat{H}\epsilon}|\varphi\rangle \ominus$

$\int d\bar{\varphi}_i d\varphi_i\, e^{-\bar{\varphi}_i\varphi_i}|\varphi_i\rangle\langle\varphi_i|$

$\langle\varphi'|\hat{H}(\hat{a}^+, \hat{a})|\varphi\rangle = H(\hat{a}^+ \mapsto \bar{\varphi}', \hat{a} \mapsto \varphi)\langle\varphi'|\varphi\rangle$

$\omega\bar{\varphi}'\varphi \cdot \langle\varphi'|\varphi\rangle$

$i \cdot i\epsilon\bar{\varphi}_k \frac{(\varphi_k - \varphi_{k-1})}{\epsilon}$ $\quad \int dt$

$\ominus \int \prod_{k=1}^{N-1} d\bar{\varphi}_k d\varphi_k \cdot \exp\left(-\sum_{k=1}^{N-1}\bar{\varphi}_k\varphi_k + \sum_{k=1}^{N}\bar{\varphi}_k\varphi_{k-1} - i\epsilon\sum_{k=1}^{N} H(\bar{\varphi}_k, \varphi_{k-1})\right) =$

$= \int \mathcal{D}[\bar{\varphi}(t), \varphi(t)] \exp\left(iS[\bar{\varphi}(t), \varphi(t)]\right)$

$$\boxed{S = \int dt \cdot \left(i\bar{\varphi}\partial_t\varphi - H(\bar{\varphi}, \varphi)\right)}$$

2) $\hat{H} = \sum_i \hat{a}_i^+\hat{a}_i \cdot E_i \Leftrightarrow \int \left(\frac{\nabla\psi^+\nabla\psi}{2m} + U\psi^+\psi\right)dx$

$\hat{\psi}(x) = \sum_i \varphi_i^{(x)} \cdot \hat{a}_i$

$$\Rightarrow \langle\Omega|T\{\hat{\psi}(\vec{x}, t)\hat{\psi}^+(\vec{y}, t')\}|\Omega\rangle = \frac{\int \mathcal{D}[\bar{\psi}(\vec{x}, t), \psi(\vec{x}, t)] \cdot \psi(\vec{x}, t)\bar{\psi}(\vec{y}, t')\, e^{iS[\bar{\psi}, \psi]}}{\int \mathcal{D}[\bar{\psi}, \psi]\, e^{iS[\bar{\psi}, \psi]}}$$

Fermi, Bose

$$H[\bar{\psi}, \psi] = \int dt\, d^d\vec{x} \left(\frac{\nabla\bar{\psi}\nabla\psi}{2m} + U\bar{\psi}\psi\right) + \frac{1}{2}\int dt\, d\vec{x}\, d\vec{x}'\, U(\vec{x} - \vec{x}')\bar{\psi}\psi(x)\, \bar{\psi}\psi(x')$$

$$S = \left[\int dt\, dx\, (i\bar{\psi}\partial_t\psi)\right] - H$$

Слабонеидеальный Бозе-Газ

N-частичн. кв. мех

$\langle\Psi|\int d^3x\, \hat{\psi}^+(x)\hat{\psi}(x)|\Psi\rangle = N$

$\Leftrightarrow$ функц. инт-ал $\int d^3x\, |\psi|^2 = N$

Осн. сост. $= \min\limits_{\langle\psi|\hat{N}|\psi\rangle = N} \langle\psi|\hat{H}|\psi\rangle = \min\limits_{\forall|\psi\rangle} (\langle H\rangle - \mu\langle N\rangle) \mapsto |\psi(\mu)\rangle$

хим. потенциал

μ найден из усл. $\langle\psi(\mu)|\hat{N}|\psi(\mu)\rangle = N$

$\hat{H} \mapsto \hat{H}' = \hat{H} - \mu\hat{N}$, но без доп. огранич.

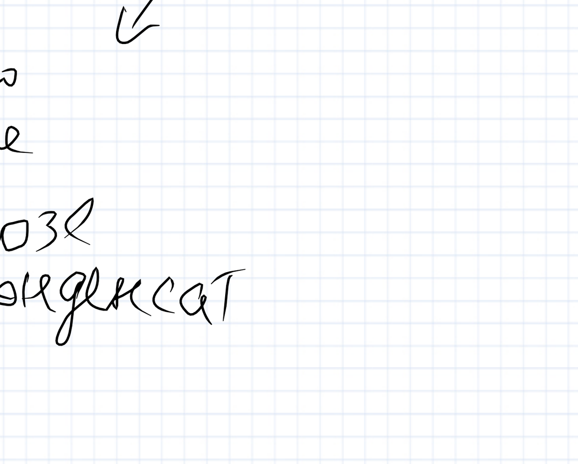
Большой каноннич. анс.

N бозе-частиц.

$U(x - y) = g\delta(\vec{x} - \vec{y})$ $\quad g$ - слабое притяж

$\hat{H}' = \int\left(\frac{\nabla\psi^+\nabla\psi}{2m} - \mu\psi^+\psi + \frac{g}{2}\psi^+\psi^+\psi\psi\right)$

$+U(\psi) = -\mu|\psi|^2 + \frac{g}{2}|\psi|^4$



$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(i\int dt\,d\vec{x}\left(i\bar{\psi}\partial_t\psi - \frac{|\nabla\psi|^2}{2m} + \mu|\psi|^2 - \frac{g}{2}|\psi|^4\right)\right)$

"метод перевала"

$\frac{\delta S}{\delta\bar{\psi}} = 0 \Rightarrow \boxed{i\partial_t\psi + \frac{\Delta\psi}{2m} + \mu\psi - g|\psi|^2\psi = 0}$ НУШ Гросса – Питаевского

$g>0$ – отталкивание

$\psi(\vec{x}, t) = \mathrm{const}$ $\qquad \mu\psi = g|\psi|^2\psi$

$\int|\psi|^2 dx = \mathrm{const}$ $\qquad \left[\begin{array}{l}\psi = 0 \\ |\psi|^2 = \mu/g\end{array}\right. \Rightarrow \psi_0 = \sqrt{\frac{\mu}{g}}e^{i\varphi}$ бозе-конденсат

$\psi = \psi_0 + \delta\psi$ $\qquad$ Спонтанное нарушение симметрии $U(1)$

$0 = \langle\Omega|\hat{\psi}(\vec{x}, t)|\Omega\rangle = \frac{\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\, \psi e^{iS}}{\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS}} = \psi_0$

$\hat{N}|\Omega\rangle = N|\Omega\rangle$

$\langle\Omega|\hat{\psi}(x, t)\hat{\psi}^+(\vec{x}', t')|\Omega\rangle = |\psi_0|^2 + \text{попр.}$

Дальный порядок $|x - \vec{x}'| \to \infty$

$|\Omega\rangle$ – когерентное сост. с

приближение $\to \hat{\psi}|\Omega\rangle = \psi_0|\Omega\rangle$

$\hat{\psi}(x)|\Omega_1\rangle = \sqrt{\frac{\mu}{g}}e^{i\varphi_1}|\Omega_1\rangle$

$\hat{\psi}(x)|\Omega_2\rangle = \sqrt{\frac{\mu}{g}}e^{i\varphi_2}|\Omega_2\rangle$

$\int|\psi|^2 dx = N$

$$\langle\Omega_1|\Omega_2\rangle = \exp\left(-\int dx\, \bar{\psi}_1\psi_2\right) = \exp\left(-\frac{\mu}{g}\cdot V e^{i(\varphi_1 - \varphi_2)}\right)$$

$$P = \frac{|\langle\Omega_1|\Omega_2\rangle|^2}{|\langle\Omega_1|\Omega_1\rangle| \cdot |\langle\Omega_2|\Omega_2\rangle|} = \exp(-2N(1 - \cos\Delta\varphi))$$

10^{23} $\qquad \Delta\varphi = O(1)$