Функциональный интеграл

$$G^R(\vec{x},\vec{y},T>0)=\langle\vec{y}|\hat{U}(T)|\vec{x}\rangle$$

$$\hat{U}(T)=\exp(-i\hat{H}T)$$

$$i\hbar\frac{\partial|\psi\rangle}{\partial t}=\hat{H}|\psi\rangle \qquad |\psi(T)\rangle=e^{-i\hat{H}T}|\psi(0)\rangle$$

$$\psi(\vec{x},T)=\int G^R(\vec{x},\vec{x}',T)\psi(\vec{x}',0)\,d\vec{x}'$$

$$\hat{H}=\hat{K}(\hat{p})+\hat{U}(\hat{x})$$

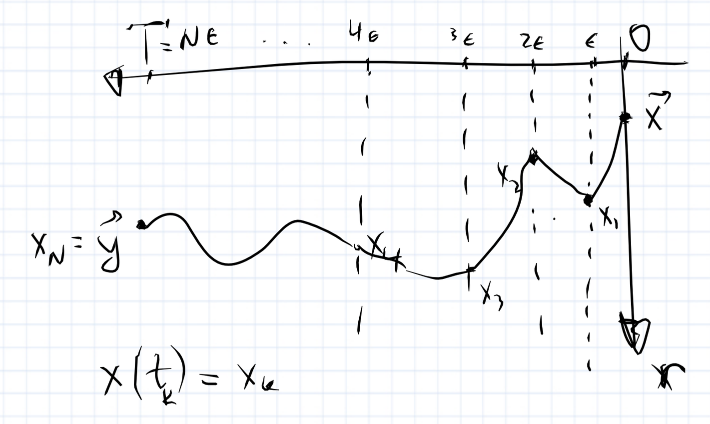
$$\hat{U}(T)=\underbrace{e^{-i\hat{H}\epsilon}e^{-i\hat{H}\epsilon}\dots e^{-i\hat{H}\epsilon}}_{N=T/\epsilon\to\infty}\ \underset{\epsilon\to 0}{\approx}$$

$$\approx e^{-i\hat{V}\epsilon}e^{-i\hat{K}\epsilon}e^{-i\hat{V}\epsilon}e^{-i\hat{K}\epsilon}\dots e^{-i\hat{V}\epsilon}e^{-i\hat{K}\epsilon}$$

$$e^{-i\hat{V}\epsilon}e^{-i\hat{K}\epsilon}\approx e^{-i\hat{H}\epsilon+\frac{1}{2}[\hat{V},\hat{K}]\epsilon^2+\dots}$$

$$G_R(\vec{x},\vec{y},T)=$$
$$=\langle\vec{x}_N=\vec{x}|e^{-i\hat{V}\epsilon}\int(d p_N)|p_N\rangle\langle p_N|e^{-i\hat{K}\epsilon}|x_{N-1}\rangle\dots\int d\vec{x}_1|\vec{x}_1\rangle\underbrace{\langle\vec{x}_1|e^{-i\hat{V}\epsilon}\int(d\vec{p}_1)|\vec{p}_1\rangle\langle\vec{p}_1|e^{-i\hat{K}\epsilon}|\vec{x}_0=\vec{x}\rangle}=$$
$$=\int\prod_{k=1}^{N-1}d\vec{x}_k\prod_{k=1}^{N}\frac{d\vec{p}_k}{(2\pi)^d}\exp\Big(\sum_{k=1}^{N}\{\underline{-iK(\vec{p}_k)\epsilon}-\underline{i\vec{p}_k\vec{x}_{k-1}}-\underline{iV(\vec{x}_k)\epsilon}+\underline{i\vec{p}_k\vec{x}_k}\}\Big)=$$

$\underline{N\to\infty,\ \epsilon\to 0}$
$$=\int\prod_{k=1}^{N-1}d\vec{x}_k\prod_{k=1}^{N}\frac{d\vec{p}_k}{(2\pi)^d}\exp\Big(i\epsilon\sum_{k=1}^{N}\Big\{\vec{p}_k\cdot\frac{\vec{x}_k-\vec{x}_{k-1}}{\epsilon}-H(\vec{p}_k,\vec{x}_k)\Big\}\Big)$$



$$x(t_k)=x_k$$

"символическая запись"

$$\prod_{k=1}^{N-1}d\vec{x}_k=\mathcal{D}x(t) \qquad \prod_{k=1}^{N}\frac{d\vec{p}_k}{(2\pi)^d}=\mathcal{D}p(t)$$

$$\begin{cases}x_0=\vec{x}\\ \vec{x}_N=y\end{cases}\to\int_{\substack{x(0)=\vec{x}\\x(T)=\vec{y}}}$$

$$G^R(\vec{x},\vec{y},T)=\int_{\substack{x(0)=\vec{x}\\ \vec{x}(t)=\vec{y}}}\mathcal{D}x(t)\mathcal{D}\vec{p}(t)\exp\Big(i\int_0^T dt\{\vec{p}\cdot\partial_t\vec{x}-H(\vec{p}(t),\vec{x}(t))\}\Big)$$
$$=\int\mathcal{D}\vec{x}(t)\mathcal{D}\vec{p}(t)\exp(i\,S[\vec{p}(t),\vec{x}(t)])$$

дифф. ф-ции из $\mathbb{R}L^2$

Мера – плоская

$$\|\{\vec{x}_k\}-\{\vec{x}'_k\}\|^2=\sum_k|\vec{x}_k-\vec{x}'_k|^2$$
$$\|\vec{x}(t)-\vec{x}'(t)\|^2=\int dt\,|\vec{x}(t)-\vec{x}'(t)|^2$$

Что можно делать?

1) Замена переменных

Пр: $H(p,x)=\frac{\vec{p}^2}{2m}+V(\vec{x})$

$$G_R=\int\mathcal{D}x\mathcal{D}p\exp\Big(i\int\Big(p\dot{x}-\frac{p^2}{2m}-U(x)\Big)dt\Big)$$
$$=\int\mathcal{D}x\,\mathcal{D}p\exp\Big(-\frac{i}{2m}\int dt\{\underbrace{p^2-2m\dot{\vec{x}}\vec{p}+(m\dot{\vec{x}})^2}_{(\vec{p}-m\dot{\vec{x}})^2}-(m\dot{\vec{x}})^2\}-i\int V dt\Big)$$

2) Взятие гауссового инт-ла

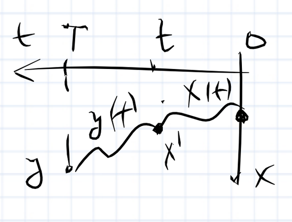
$$p'(t)=p(t)-m\dot{x}(t)$$
$$=\int\mathcal{D}x(t)\exp\Big(-i\int_0^T V(x(t))dt+i\int_0^T\frac{m\dot{x}^2}{2}dt\Big)\int\mathcal{D}p'(t)\exp\Big(-\frac{i}{2m}\int_0^T p'^2dt\Big)$$

"Взяли" Гауссов интеграл "= 1"

$$=\int\mathcal{D}x(t)\exp(i\,S[\vec{x}(t),\dot{\vec{x}}(t)])$$

дискр. $$=\Big(\frac{m}{2\pi i\epsilon}\Big)^{Nd/2}\int\prod_{k=1}^{N-1}d\vec{x}_k\exp\Big(i\epsilon\sum_{k=1}^{N}\Big(\frac{m}{2}\Big(\frac{\vec{x}_k-\vec{x}_{k-1}}{\epsilon}\Big)^2-V(\vec{x}_k)\Big)\Big)$$

3) Объединение.



$$\int dx'\int_{\substack{x(0)=x\\x(t)=x'}}\mathcal{D}x(t)\exp\Big(i\int_0^t L(t)dt\Big)\times$$
$$\times\int_{\substack{y(t)=x'\\y(T)=y}}\mathcal{D}\bar{y}(t)\exp\Big(i\int_t^T L(t)dt\Big)$$

$$\equiv\int_{\substack{x(0)=x\\x(T)=y}}\mathcal{D}x(t)\exp\Big(i\int_0^T L(t)dt\Big)$$

$$\int dx'\langle x'|\hat{U}(t)|x\rangle\times\langle y|\hat{U}(T-t)|x'\rangle$$
$$=\langle y|\hat{U}(T-t)\hat{U}(t)|x\rangle=\langle y|\hat{U}(T)|x\rangle$$

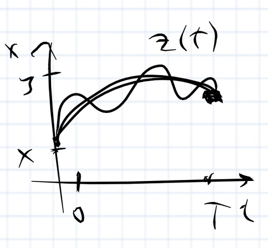
Как считать?

$$\hat{H}=\frac{\hat{p}^2}{2m}+\frac{m\omega^2\hat{x}^2}{2}$$

Трюк #1:

$$G^R(x,y,T)=\int_{\substack{x(0)=x\\x(T)=y}}\exp\Big(\frac{im}{2}\int_0^T(\dot{x}^2-\omega^2x^2)\,dt\Big)\mathcal{D}x(t)$$

$$\int dt\,\frac{i}{2}\,x(t)\hat{L}\,x(t)$$
$$\hat{L}=m(-\partial_t^2-\omega^2)$$



Ур-ние движения (классич.):

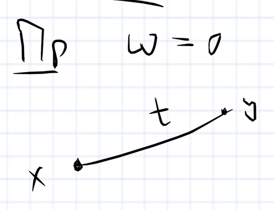
$$\frac{\delta S}{\delta x(t)}=0\iff\underline{\hat{L}\,x_{cl}(t)=0}\ +\begin{cases}x_{cl}(0)=x\\x_{cl}(T)=y\end{cases}$$

$$x(t)\Rightarrow z(t)+x_{cl}(t)$$

$$=\int_{z(0)=z(T)=0}\mathcal{D}z(t)\exp(i\,S[z(t)]+i\,S_{cl})$$

"Метод перевала": $$=\exp(iS_{cl}(\vec{x},\vec{y},T))\int_{z(0)=z(T)=0}\mathcal{D}z(t)\exp(iS[z])$$

Пр $\omega=0$



$$\dot{x}_{cl}(t)=\text{const}=\frac{y-x}{T}$$
$$x_{cl}(t)=x+(y-x)\frac{t}{T}$$

$$S_{cl}=T\cdot\frac{m}{2}\frac{(y-x)^2}{T^2}=\frac{m(y-x)^2}{2T}$$

$$\boxed{G^R(x,y,T)=G^R(0,0,T)\exp(iS_{cl})}$$

$$G^R(x,y,T)=\sqrt{\frac{m}{2\pi iT}}\exp\Big(\frac{im}{2T}(x-y)^2\Big)$$

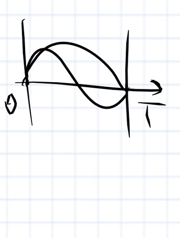
Вернёмся к осц.

Трюк #2: Отношение ф.и. – хорошо определено $\hat{L}_\omega=m(-\partial_t^2-\omega^2)$

$$\frac{G^R_\omega(0,0,T)}{G^R_0(0,0,T)}=\frac{\int\mathcal{D}z(t)\exp\big(\frac{i}{2}\int dt\,z\hat{L}_\omega z\big)}{\int\mathcal{D}z(t)\exp\big(\frac{i}{2}\int dt\,z\hat{L}_0z\big)}$$

Трюк #3

1) Собств. значение:
$$\begin{cases}\hat{L}_\omega z_n(t)=\lambda_n z_n(t)\\ z_n(0)=0=z_n(T)\end{cases}$$



$$-m(\ddot{z}_n+\omega^2z_n)=\lambda_n z_n(t)$$

$$z_n(t)=\sqrt{\frac{2}{T}}\sin\Big(\frac{\pi nt}{T}\Big)\qquad n=1,\dots,\infty$$

$$\int_0^T z_n(t)z_m(t)dt=\delta_{nm}$$

$dz_1\dots dz_N$

$$\lambda_n=m\Big(\frac{\pi^2n^2}{T^2}-\omega^2\Big)$$

2) $$\underline{z(t)=\sum_n c_nz_n(t)}\longrightarrow\exp\Big(\frac{i}{2}\sum_n c_n^2\lambda_n\Big)$$

$$\mathcal{D}z(t)=\prod_{n=1}^{\infty}dc_n$$ ← ортогональное преобразование ($J=1$)

$$\|z-z'\|^2=\int_0^T|z(t)-z'(t)|^2dt$$

ортонорм.

$$=\int_0^T\Big|\sum_n(c_n-c'_n)z_n(t)\Big|^2dt=$$
$$=\sum_n(c_n-c'_n)^2\Rightarrow g=\hat{1},\ J=1$$

$$\sqrt{\frac{\det(\hat{L}_0)}{\det(\hat{L}_\omega)}}=\frac{G^R_\omega(0,0,T)}{G^R_0(0,0,T)}=\frac{\int\prod_n dc_n\exp\big(\frac{i}{2}\sum_n c_n^2\lambda_n(\omega)\big)}{\int\prod_n dc_n\exp\big(\frac{i}{2}\sum_n c_n^2\lambda_n(0)\big)}=\frac{\prod_{n=1}^{\infty}\sqrt{\frac{2\pi i}{\lambda_n(\omega)}}}{\prod_{n=1}^{\infty}\sqrt{\frac{2\pi i}{\lambda_n(0)}}}=\prod_{n=1}^{\infty}\sqrt{\frac{\lambda_n(0)}{\lambda_n(\omega)}}$$

$$=\Big(\prod_{n=1}^{\infty}\Big(1-\frac{\omega^2T^2}{\pi^2n^2}\Big)\Big)^{-1/2}=$$ ← Хорошо определено

$$=\exp\Big(-\frac{1}{2}\sum_{n=1}^{\infty}\ln\Big(1-\frac{\omega^2T^2}{\pi^2n^2}\Big)\Big)=$$

$f(\omega T)$

$f'(\omega T)|=\sum$ дробо-раци.

полюса $z_n=\pm\pi n$
вычеты C_n

$$=\sqrt{\frac{\omega T}{\sin(\omega T)}}$$

$$\frac{G^R_\omega(0,0,T)}{G^R_0(0,0,T)}=\sqrt{\frac{\omega T}{\sin\omega T}}\Rightarrow G^R_\omega(0,0,T)=\sqrt{\frac{m\omega}{2\pi i\sin\omega T}}=\sqrt{\frac{m\omega}{\pi(e^{i\omega T}-e^{-i\omega T})}}$$
$$=\sqrt{\frac{m\omega}{\pi}}e^{-i\omega T/2}\frac{1}{\sqrt{1-e^{-2i\omega T}}}=$$
$$=\sqrt{\frac{m\omega}{\pi}}e^{-i\omega T/2}\sum_{n=0}^{\infty}C^n_{-1/2}(-1)^ne^{-2in\omega T}$$

$$E_n=\omega(2n+1/2)$$

Спектр

$$G^R(x,y,T)=\theta(T)\langle y|e^{-i\hat{H}T}|x\rangle$$
$$=\sum_n\theta(T)\langle y|n\rangle e^{-iE_n(T)}\langle n|x\rangle$$
$$=\sum_n\theta(T)\,\psi_n(y)\psi_n^*(x)e^{-iE_nT}$$

$$\mathrm{Tr}\,\hat{G}^R(T)=\int dx\,G^R(x,x,T)=\sum_n\theta(T)e^{-iE_nT}\to\lim_{\epsilon\to 0}\int_0^{\infty}e^{i\omega T}e^{-iE_nT}e^{-\epsilon t}dt$$

$$\mathrm{Tr}(\hat{G}^R(\omega))=\sum_n\frac{i}{\omega-E_n+i0}\qquad\frac{1}{x+i0}=\mathrm{v.p.}\frac{1}{x}-i\pi\delta(x)$$

$$\nu(\omega)=\sum\delta(\omega-\omega_n)=\mathrm{Tr}\,\mathrm{Im}(i\hat{G}^R(\omega))$$