Формализм Гельфанда-Яглома

$$\frac{\det(\hat{H}_s)}{\det(\hat{H}_0)} = \prod_n \left(\frac{\lambda_n^{(s)}}{\lambda_n^{(0)}}\right) \qquad H_s = -\partial_x^2 + \frac{s(s+1)}{ch^2 x} \qquad H_0 = -\partial_x^2$$

I способ (нулевые гран-условия)

$\epsilon = L/N$, $0 \ldots L \to x$

$$\hat{H} = -\partial_x^2 + U(x) \qquad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{N-1} \end{pmatrix}$$

H — мат $(N-1)\times(N-1)$

$$(\hat{H}\psi)_k = -\frac{\psi_{k-1} - 2\psi_k + \psi_{k+1}}{\epsilon^2} + U(x_k)\psi_k$$

если период. Г.У.

$$\hat{H} = \begin{pmatrix} U(x_1) + \frac{2}{\epsilon^2} & -\frac{1}{\epsilon^2} & & \# \\ -\frac{1}{\epsilon^2} & U(x_2) + \frac{2}{\epsilon^2} & -1/\epsilon^2 & \\ & -1/\epsilon^2 & \ddots & \ddots \\ \# & & \ddots & U(x_{N-1}) + \frac{2}{\epsilon^2} \end{pmatrix} = \frac{1}{\epsilon^2} \underbrace{\begin{pmatrix} \epsilon^2 U(x_1) + 2 & -1 & & \\ -1 & \epsilon^2 U(x_2) + 2 & -1 & \\ & & -1 & \\ & & -1 & \epsilon^2 \ldots \end{pmatrix}}_{f_{N-1} = \det(\ldots)}$$

$\psi_0 \equiv 0$

$\psi_N \equiv 0$

$$\det \hat{H} = \frac{1}{\epsilon^{2(N-1)}} f_{N-1}$$

$$f_{N-1} = (U(x_{N-1})\epsilon^2 + 2) f_{N-2} - (-1)\times(-1) f_{N-3}$$

$$f_{N-1} + f_{N-3} - 2f_{N-2} = \epsilon^2 U(x_{N-1}) f_{N-2}$$

$$-\frac{f_{N-1} - 2f_{N-2} + f_{N-3}}{\epsilon^2} + U(x_{N-1}) f_{N-2} = 0$$

$$f_0 = 0$$

$$f_1 = 2 + \epsilon^2 U(x_1)$$

$$\Downarrow$$

$$f'(0) = \frac{f_1 - f_0}{\epsilon} = \frac{2}{\epsilon}$$

$\epsilon \to 0$; $f_k \equiv f(k\cdot\epsilon) \Rightarrow -f''_{(x)} + U(x) f(x) = 0$

$$g(x) = \frac{\epsilon}{2} f(x) \Rightarrow \begin{cases} g(0) = 0 \\ g'(0) = 1 \end{cases}$$

1) $\det \equiv f(L) = \frac{\epsilon}{2} g(L)$

2) $\begin{cases} g(0) = 0 \\ g'(0) = 1 \end{cases} \Rightarrow \det \hat{H} = \frac{1}{2\epsilon^{2N-3}} g(L) \Rightarrow$

$$\frac{\det(H_1)}{\det(H_2)} = \frac{\det(-\partial_x^2 + U_1(x))}{\det(-\partial_x^2 + U_2(x))} = \frac{g_1(L)}{g_2(L)}$$

одинаковые гран. условия

(одинаковые асимптотики при $x \to 0$)

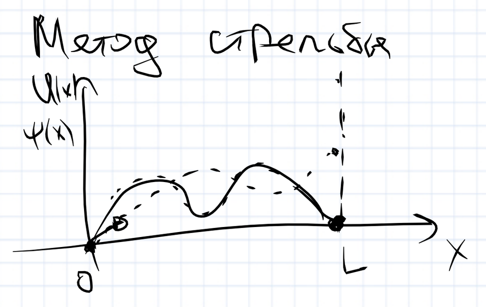
II способ

$$\frac{\det(H_1 - E)}{\det(H_2 - E)} = \frac{f_1(E)}{f_2(E)} \qquad f_i(E) = \prod_n (\lambda_n^{(i)} - E)$$

$f_i(E) = 0 \Leftrightarrow E$ — собств. число

Придумать функцию, к-я равна $0 \Leftrightarrow$ собств. число

Метод стрельбы



$\to f(0) = 0$

$f'(0) = 1$

и следим $f(L|E)$

в тот момент когда $f(L|E_n) = 0$

$f(x|E_n)$ — собств. ф-ция

$f(L|E) \equiv g(E)$ — характеристический полином

(с точностью до целой ф-ции)

Если $f_i(x \to 0)$ одинак. асимпт. $\Rightarrow$ целые ф-ции будут одинаковые

Примеры

1) Квантовый осциллятор $\quad S[x] = \int_0^\beta d\tau \left((\partial_\tau x)^2 + \omega^2 x^2(\tau)\right)$

$$\frac{\det(\hat{H}_\omega)}{\det(\hat{H}_0)} = \frac{sh(\omega\beta)/\omega}{\beta} = \frac{sh(\omega\beta)}{\omega\beta} \qquad \hat{H}_\omega = -\partial_\tau^2 + \omega^2$$

$$(-\partial_\tau^2 + \omega^2) f(\tau) = 0$$

$$f(\tau) = C_1 ch(\omega\tau) + C_2 sh(\omega\tau)$$

$$-\partial_\tau^2 f_2(\tau) = 0$$

$$f(0) = 0 \Rightarrow C_1 = 0$$

$$f_2(\tau) = C_1 + C_2 \tau = \tau$$

$$f(\tau \to 0) = \tau \Rightarrow C_2 = \frac{1}{\omega}$$

$$\Rightarrow f(\tau) = \frac{sh(\omega\tau)}{\omega}$$

2) Инстантон. "$E = -\varkappa^2$"

$$\frac{\det(H_s + \varkappa^2)}{\det(H_0 + \varkappa^2)} = ? \qquad H_s = -\partial_\tau^2 + \frac{s(s+1)}{ch^2 \tau}$$

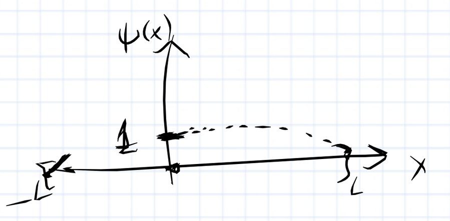
Для двух-ямного пот. $s = 2$

1) $L \to \infty \Rightarrow$ поместить в ящик $2L \to \infty$

2) Потенциал чётный

$$\hat{H} = \left(\begin{array}{c|c} \text{чётн.} & 0 \\ \hline 0 & \text{нечёт.} \end{array}\right)$$

$$\Rightarrow \det \hat{H} = \det(\hat{H}^{(+)}) \det(\hat{H}^{(-)})$$



$\det(H^{(+)}) = \psi^{(+)}(L) \leftarrow$ обобщить ГЯ $\to$ чётн.

$$\begin{cases} \psi_+(0) = 1 \\ \psi'_+(0) = 0 \end{cases}$$

$$\det(H^{(-)}) = \psi_-(L)$$

$$\begin{cases} \psi_-(0) = 0 \\ \psi'_-(0) = 1 \end{cases}$$

$x \leftrightarrow \tau$

$$-\psi'' + \frac{s(s+1)\psi(x)}{ch^2 x} = -\varkappa^2 \psi(x)$$

$$y = ch^2 x \Rightarrow y(1-y)\psi''(y) + \left(\tfrac{1}{2} - y\right)\psi'(y) - \left(\frac{s(s+1)}{4y} - \frac{\varkappa^2}{4}\right)\psi(y) = 0$$

$$\psi(y) = y^{(s+1)/2} \chi(y) \Rightarrow y(1-y)\chi''(y) + \left[(s + 3/2) - y(s+2)\right]\chi'(y) - \frac{(s+1)^2 - \varkappa^2}{4}\chi(y) = 0$$

$$z = 1 - y = -sh^2 x \Rightarrow z(1-z)\chi''(z) + \left(\tfrac{1}{2} - z(s+2)\right)\chi'(z) - \frac{(s+1)^2 - \varkappa^2}{4}\chi(z) = 0$$

$$\begin{cases} a = \frac{s+1+\varkappa}{2} \\ b = \frac{s+1-\varkappa}{2} \\ c = 1/2 \end{cases} \Rightarrow \quad {}_2F_1(a, b; c; z) \sim z^{1-c}\, {}_2F_1(b-c+1, a-c+1; 2-c, z)$$

$$\begin{cases} \psi^{(+)}(x) = ch^{s+1} x \cdot {}_2F_1\left(\frac{s+1+\varkappa}{2}, \frac{s+1-\varkappa}{2}; \frac{1}{2}; -sh^2 x\right) \\ \psi^{(-)}(x) = sh\, x \cdot ch^{s+1} x \cdot {}_2F_1\left(\frac{s-\varkappa+2}{2}, \frac{s+\varkappa+2}{2}; \frac{3}{2}; -sh^2 x\right) \end{cases}$$

$\approx 1 - \# x^2$

$E = -\varkappa^2$

$$\Rightarrow \frac{\det(H_s + \varkappa^2)}{\det(H_0 + \varkappa^2)} = \frac{\psi_s^{(+)}(L)\psi_s^{(-)}(L)}{\psi_0^{(+)}(L)\psi_0^{(-)}(L)} \quad (L \to \infty)$$

$$\begin{cases} \psi^{(+)}(x) = \dfrac{\Gamma(1/2)\Gamma(\varkappa)}{2^\varkappa \Gamma\left(\frac{1+s+\varkappa}{2}\right)\Gamma\left(\frac{\varkappa-s}{2}\right)} e^{\varkappa x} \\ \psi^{(-)}(x) = \dfrac{\Gamma(3/2)\Gamma(\varkappa)}{2^\varkappa \Gamma\left(\frac{s+\varkappa+2}{2}\right)\Gamma\left(\frac{1-s+\varkappa}{2}\right)} e^{\varkappa x} \end{cases} \quad x \to \infty$$

$$\begin{cases} \psi_0^{(+)} = ch\,\varkappa x \sim \frac{1}{2} e^{\varkappa x} \\ \psi_0^{(-)} = \frac{sh(\varkappa x)}{\varkappa} \sim \frac{1}{2\varkappa} e^{\varkappa x} \end{cases}$$

$$\boxed{\frac{\det(H_s + \varkappa^2)}{\det(H_0 + \varkappa^2)} = \frac{\varkappa\, \Gamma^2(\varkappa)}{\Gamma(1+s+\varkappa)\Gamma(\varkappa - s)}}$$

$z = \frac{\omega\tau}{2}$

Применяем

без нулевой моды!

$$\frac{\det'\left(-\partial_\tau^2 + \omega^2 - \frac{3\omega^2}{2ch^2 \frac{\omega\tau}{2}}\right)}{\det'(-\partial_\tau^2 + \omega^2)} = \frac{\det\left(-\partial_z^2 + 4 - \frac{6}{ch^2 x}\right)}{\det(-\partial_z^2 + 4)}$$

Нулевая мода!

$\varkappa = 2$; $s = 2$

$\varkappa^2 = 4 + \epsilon \Rightarrow$ нулевая мода $\to \epsilon$

$\Gamma(x) \sim \frac{1}{x}$

$$\det'(H_s + \varkappa^2) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \cdot \det(H_s + \varkappa^2 + \epsilon)$$

$$\varkappa = 2 + \frac{\epsilon}{4} \Rightarrow \frac{\det'(\ldots)}{\det'(\ldots)} = \lim_{\epsilon \to 0} \frac{\frac{1}{\epsilon}\det(\ldots)_{+\epsilon}}{\frac{1}{4}\det(\ldots)_{+\epsilon}} = \frac{2\Gamma^2(2)}{\Gamma(5)\cdot\Gamma\left(\frac{\epsilon}{4}\right)} \cdot \frac{4}{\epsilon} = \frac{2}{24} = \boxed{\frac{1}{12}}$$